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**Explicitly Solvable Complex Chebyshev Approximation
Problems Related to Sine Polynomials**

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Abstract

Explicitly solvable real Chebyshev approximation problems on the unit interval are typically characterized by simple error curves. In this note, we present a similar principle for complex approximation problems with error curves induced by sine polynomials. As an application, some new explicit formulae for complex best approximations are derived.

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§1. Introduction

We consider complex Chebyshev approximation problems on the unit interval $I = [-1, 1]$ of the type

$$E_n(f; w) := \min_{p \in \Pi_n} \|f - p\|_w, \quad \|g\|_w := \max_{z \in I} |w(z)g(z)|. \quad (1)$$

Here, $w > 0$ is a continuous weight function on I and Π_n denotes the set of complex polynomials of degree at most n . There are a number of classical real problems of the form (1) for which best approximations are explicitly known (see e.g. the books of Achieser [1] and Meinardus [6]). As Gutknecht [5] pointed out, there is a unified treatment for almost all these examples. Typically, by means of the Joukowski map, the error function for (1) can be written as the real part of a corresponding circular error function on the unit circle which has sufficiently large winding number so that the alternation criterion for (1) is fulfilled.

Examples for explicitly solvable complex problems (1) are more scarce. Recently, Freund and Ruscheweyh [2] and Freund [3,4] found complex best approximations for certain cases which all involve a purely imaginary parameter. Naturally, the question arises whether there is — similar as in the real case — a more general principle for explicitly solvable complex problems which includes the cases treated in [2,3,4] and explains the special role of the purely imaginary parameter.

In this note, we give a positive answer to this question. Section 2 contains our main result which states that complex error curves induced by complex sine polynomials always guarantee optimality for (1). Some applications of this result are given in Section 3.

§2. A sufficient condition for complex best approximations

In the sequel, the following notations are used. $p^*(v) := v^n \bar{p}(1/v)$ denotes the reciprocal to the polynomial p of degree n . We call $p(v) = v^n + \dots \in \Pi_n$ a *sine* (resp. *cosine*) *polynomial* if all the zeros of p are contained in $|v| < 1$ and

$$|p(e^{i\varphi})|^2 = \alpha_0 + \sum_{j=1}^n \alpha_j \sin j\varphi \quad (\text{resp. } |p(e^{i\varphi})|^2 = \sum_{j=0}^n \beta_j \cos j\varphi), \quad \varphi \in \mathbb{R}.$$

Note that cosine polynomials have real coefficients, while sine polynomials are in general complex.

Theorem 1. *Let h and p_c be cosine polynomials of degree r and c , respectively, p_s be a sine polynomial, and set*

$$e(v) = \frac{h(v)}{|h(v)|} \frac{p_c(v)}{p_c^*(v)} |p_s(v)|^2. \quad (2)$$

Let $p \in \Pi_n$ be such that for some constant $\sigma \in \mathbb{C}$

$$w(z)(f(z) - p(z)) = \sigma \frac{1}{2} \left(e(v) + c\left(\frac{1}{v}\right) \right), \quad z = \frac{1}{2} \left(v + \frac{1}{v} \right), \quad |v| = 1. \quad (3)$$

Then, if $r + c \geq n + 1$, p is the unique best approximation in the $\|\cdot\|_w$ -norm to f out of Π_n , and

$$E_n(f; w) = |\sigma| |p_s(1)|^2.$$

Proof: For $\varphi \in \mathbb{R}$, we set

$$\alpha(\varphi) = |p_s(e^{i\varphi})|^2, \quad \gamma(\varphi) = \arg \frac{h(e^{i\varphi})}{|h(e^{i\varphi})|} \frac{p_c(e^{i\varphi})}{p_c^*(e^{i\varphi})}.$$

The assumptions on p_s , h , p_c imply that $\alpha(\varphi) + \alpha(-\varphi) = 2|p_s(1)|^2$ and that γ is a continuous nondecreasing function with $\gamma(\pi) - \gamma(0) = (r + c)\pi$. Next, define $g(\varphi) := w(z)(f(z) - p(z))$, $z = \cos \varphi$, $0 \leq \varphi \leq \pi$. With (2), (3) it follows that

$$g(\varphi) = \frac{\sigma}{2}(e^{i\gamma(\varphi)}\alpha(\varphi) + e^{-i\gamma(\varphi)}\alpha(-\varphi)) \quad \text{and} \quad |g(\varphi)| \leq |\sigma| |p_s(1)|^2$$

where equality holds for all φ with $e^{i\gamma(\varphi)} = \pm 1$. There are at least $r + c + 1$ such points in $[0, \pi]$:

$$\varphi_0 = 0 < \varphi_1 < \dots < \varphi_{r+c} = \pi \quad \text{and} \quad g(\varphi_j) = (-1)^j \sigma |p_s(1)|^2, \quad 0 \leq j \leq r+c.$$

Hence, $\|f - p\|_w$ is attained in at least $r + c + 1 \geq n + 2$ successive points in $[-1, 1]$, with alternating signs, and this guarantees (e.g. [6,8]) that p is the unique best approximation to f . ■

Note that for $p_s \equiv \text{const}$, Theorem 1 reduces to the result on real approximation problems (1) mentioned in the introduction. The complex best approximations obtained in [2,3,4] have error curves of the form (3) with suitable sine polynomials of degree 1. However, all linear sine polynomials are given by $p_s(v) = v - i\xi$ with ξ purely imaginary and $|\xi| < 1$, and this explains the purely imaginary parameter in the examples of [2,3,4].

Theorem 1 can be used to derive new explicit formulae for complex best approximations in a number of cases. Two typical examples are given in the next section.

§3. Examples

First, we consider (1) with weight functions of the type $w(z) = 1/\sqrt{\rho(z)}$ where $\rho > 0$ on I is a polynomial of degree r . For any integer j , set

$$T_j(z; \rho) = \frac{1}{2}\sqrt{\rho_0} \left(v^{j-r} h(v) + \frac{1}{v^{j-r}} h\left(\frac{1}{v}\right) \right), \quad (4)$$

where $\rho_0 > 0$ and h is a cosine polynomial defined by

$$\rho(z) = \rho_0 h(v) h(1/v) \quad (5)$$

(cf. Achieser [1, p. 249]). Here and in the sequel, it is always assumed that z and v are connected by the Joukowski map

$$z = z(v) = \frac{1}{2} \left(v + \frac{1}{v} \right).$$

Note that $T_j(z; \rho)$ is a polynomial in z of degree $\max\{j, r-j\}$ and, moreover, for $\rho \equiv 1$, $T_j(z) := T_j(z; \rho)$ is the usual Chebyshev polynomial of degree $|j|$.

Theorem 2. Let $l \geq 1$ and m be integers, $\xi = it$, $-1 < t < 1$, and

$$f(z) = \frac{T_m(z; \rho) - \xi T_{m-l}(z; \rho)}{1 + \xi^2 - 2\xi T_l(z)} \quad \left(= \sum_{j=0}^{\infty} \xi^j T_{lj+m}(z; \rho) \right) \quad (6)$$

Let $n, k \geq 0$ be integers which satisfy

$$lk + m \leq n < l(k+1) + m \quad \text{and} \quad r/2 \leq l(k-1) + m \quad (7)$$

Then,

$$\begin{aligned} p(z) = & \sum_{j=0}^k \xi^j T_{lj+m}(z; \rho) - \frac{\xi^{k+2}}{(1 - \xi^4)(1 - \xi^2)} \left(\xi^2 T_{l(k-2)+m}(z; \rho) \right. \\ & \left. - \xi(1 - \xi^2) T_{l(k-1)+m}(z; \rho) + (\xi^4 - \xi^2 - 1) T_{lk+m}(z; \rho) \right) \end{aligned} \quad (8)$$

is the unique best approximation in the $\|\cdot\|_w$ -norm to f out of Π_n , and

$$E_n(f; w) = |t|^{k+1} / (1 - t^4) \quad .$$

Proof: First, note that, in view of (7),

$$\text{degree } p = \max \{lk + m, r - l(k-2) - m\} = lk + m \leq n \quad ,$$

and hence $p \in \Pi_n$. Next, set

$$F(v) = \sqrt{\rho_0} h(v) v^{m-r} \sum_{j=0}^{\infty} \xi^j v^{lj} = \sqrt{\rho_0} h(v) \frac{v^{m-r}}{1 - \xi v^l} \quad (9)$$

and

$$\begin{aligned} P(v) = & \sqrt{\rho_0} h(v) v^{m-r} \left(\sum_{j=0}^k \xi^j v^{lj} - \sigma \xi v^{l(k-2)} \left(\xi^2 - \xi(1 - \xi^2) v^l \right. \right. \\ & \left. \left. + (\xi^4 - \xi^2 - 1) v^{2l} \right) \right) , \quad \sigma := \frac{\xi^{k+1}}{(1 - \xi^4)(1 - \xi^2)} . \end{aligned} \quad (10)$$

Using (4), (6), and (8), one easily verifies that

$$f(z) = \frac{1}{2} (F(v) + F(\frac{1}{v})) \quad , \quad p(z) = \frac{1}{2} (P(v) + P(\frac{1}{v})) \quad . \quad (11)$$

Moreover, from (9), rewritten in the form

$$F(v) = \sqrt{\rho_0} h(v) v^{m-r} \left(\sum_{j=0}^k \xi^j v^{lj} + \frac{\xi^{k+1} v^{l(k+1)}}{1 - \xi v^l} \right) \quad ,$$

and (10), we deduce that

$$F(v) - P(v) = \sigma \sqrt{\rho_0} h(v) v^{l(k-2)+m-r} (v^l - \xi)^2 \frac{v^l + \xi}{1 - v^l \xi} \quad (12)$$

Finally, note that (5) implies

$$w(z)h(v) = w(z)h(1/v) = \frac{1}{\sqrt{\rho_0}} \frac{h(v)}{|h(v)|} = \frac{1}{\sqrt{\rho_0}} \frac{h(1/v)}{|h(1/v)|} \quad \text{if } |v| = 1 \quad ,$$

and, together with (11), (12), and the fact that

$$(v^l - \xi)(1 + v^l \xi) = v^l |v^l - \xi|^2 \quad \text{if } |v| = 1 \quad ,$$

it follows that the error curve associated with p is of the form (3), (2) with

$$p_c(v) = v^{l(k-1)+m-r} (v^{2l} - \xi^2) \quad , \quad p_s(v) = v^l - \xi \quad .$$

Clearly, p_c and p_s are cosine and sine polynomials, respectively. (7) guarantees that

$$r + \text{degree } p_c = l(k+1) + m \geq n+1 \quad ,$$

and, by Theorem 1, p is the best approximation to f . ■

We remark that this theorem is an extension of a result for the case $\rho \equiv 1$ due to Rivlin [7] who showed that the best approximations to real functions of the form (6) with $-1 < \xi < 1$ are given by

$$p(z) = \sum_{j=0}^k \xi^j T_{lj+m}(z) + \frac{\xi^{k+2}}{1 - \xi^2} T_{lk+m}(z) \quad .$$

In the rest of this paper, only uniform approximation $w \equiv \rho \equiv 1$ is considered, and $\| \cdot \|$ will denote the uniform norm on I . We set

$$f_a(z) = \frac{1}{z - a} \quad \text{where } a = \frac{1}{2}(\xi + \frac{1}{\xi}) \quad , \quad \xi = it \quad , \quad -1 < t < 1 \quad , \quad t \neq 0 \quad .$$

Then,

$$f_a(z) = \frac{4\xi}{\xi^2 - 1} \left(f(z) - \frac{1}{2} \right)$$

where f is defined by (6) with $m = 0$, $l = 1$, $\rho \equiv 1$, and we obtain the following corollary to Theorem 2.

Corollary. Let $n \in \mathbb{N}$ and p be defined by (8) with $k = n$, $m = 0$, $l = 1$, $\rho \equiv 1$. Then,

$$p_n(z) = \frac{4\xi}{\xi^2 - 1} \left(p(z) - \frac{1}{2} \right) \quad (13)$$

is the best uniform approximation to f_a out of Π_n and

$$\|f_a - p_n\| = \frac{4|t|^{n+2}}{(1-t^4)(t+t^2)} \quad (14)$$

In [4], we derived the formula (14) for the degree of approximation in a different manner. The explicit representation (13), (8) of the optimal polynomial is new.

A well-known sufficient condition (see e.g. [9]) for best uniform approximation on the unit disk states that optimality is guaranteed if the error function is a Blaschke product b with sufficiently large winding number. Our final example shows how complex best approximations on the unit interval can be derived if b is induced by the product of a sine and cosine polynomials.

Theorem 3. Let $\sigma \in \mathbb{C}$, $P \in \Pi_n$, and F be such that

$$F(v) - P(v) = \sigma \frac{q_c(v)}{q_c^*(v)} \frac{q_s(v)}{q_s^*(v)} \quad (15)$$

holds with cosine and sine polynomials q_c, q_s of degrees c, s . Set

$$f(z) = \frac{1}{2} \left(F(v) \bar{q}_s^2(v) + F\left(\frac{1}{v}\right) \bar{q}_s^2\left(\frac{1}{v}\right) \right), \quad p(z) = \frac{1}{2} \left(P(v) \bar{q}_s^2(v) + P\left(\frac{1}{v}\right) \bar{q}_s^2\left(\frac{1}{v}\right) \right). \quad (16)$$

Then, if $c + s \geq n + 1$, p is the best uniform approximation to f out of Π_{n+2s} on I and

$$\|f - p\| = |\sigma| |q_s(1)|^2$$

Proof: Obviously, $p \in \Pi_{n+2s}$. Next, note that

$$\bar{q}_s^2(v) = v^s \frac{\bar{q}_s(v)}{\bar{q}_s^*(v)} |\bar{q}_s(v)|^2 \quad \text{if } |v| = 1,$$

and together with (15), (16) it follows that the error curve $f(z) - p(z)$, $z \in I$, is of the form (3), (2) with

$$h \equiv 1, \quad p_c(v) \equiv v^s q_c(v) q_s(v) \bar{q}_s(v), \quad p_s(v) = \bar{q}_s(v).$$

p_c is a cosine polynomial of degree $c + 3s$ ($\geq n + 2s + 1$), and with q_s also \bar{q}_s is a sine polynomial. Hence Theorem 1 implies that p is the best approximation to f . ■

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